Viette’s formulas relate the coefficients of a polynomial to various sums and products of its roots.

NOTE: Viette’s formulas refer to all roots, including Complex, NOT just the Real roots.

EX. 1: \( a_n = 1 \): Expand and simplify each into the form: \( F(x) = x^n \pm x^{n-1} \pm x^{n-2} \pm \ldots \pm x \pm \) ________

\( F_2(x) = (x - r_1)(x - r_2) \)
\( F_3(x) = (x - r_1)(x - r_2)(x - r_3) \)
\( F_4(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4) \)

EXERCISE 1: Using the above patterns, each of these is a one-step question. Be careful of the ± signs.

For each polynomial determine: 1. sum of roots; 2. product of roots; 3. sum of all “product pairs” of roots.

A. \( F(x) = x^4 + 7x^3 + 5x^2 + x + 9 \)  __________; __________; __________

B. \( G(x) = x^3 - 9x^2 + 3x - 4 \) __________; __________; __________

C. \( H(x) = x^4 - 2x^2 - x - 10 \) __________; __________; __________

LEMMA Let \( a_n = 1 \), IF \( F(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \), then COMPLETE:

The SUM of the roots of \( F(x) = r_1 + r_2 + r_3 \ldots = \) __________

AND the SUM of the all ‘Product Pairs’ of the roots of \( F(x) = r_1 r_2 + r_1 r_3 + r_2 r_3 + \ldots = \) __________

AND the PRODUCT of the roots of \( F(x) = r_1 r_2 r_3 \ldots = \) __________

THEOREM – Viete’s (16th C) Formulas IF \( F(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \), then COMPLETE:

The SUM of the roots of \( F(x) = r_1 + r_2 + r_3 \ldots = \) ______________

AND the SUM of the all ‘Product Pair’s’ of the roots of \( F(x) = r_1 r_2 + r_1 r_3 + r_2 r_3 + \ldots = \) __________

AND the PRODUCT of the roots of \( F(x) = r_1 r_2 r_3 \ldots = \) ______________

EXERCISE 2: Using the above patterns, each of these is a one-step question. Be careful of the ± signs.

For each polynomial determine: 1. sum of roots; 2. product of roots; 3. sum of all “product pairs” of roots.

A. \( F(x) = 2x^4 + 7x^3 + 5x^2 + x + 9 \) ________; ________; ________

B. \( G(x) = -4x^3 + 9x^2 + 3x - 4 \) ________; ________; ________

EXERCISE 3:

A. The four roots of \( F(x) = x^4 + Ax^3 + Bx^2 + Cx + D \) are 1, -4, 5, and 5. Compute \( C = \) ________

B. Two of the roots of \( G(x) = 2x^3 + Ax^2 + Bx + C \) are \( \frac{5 - \sqrt{7}}{2} \) and \(-7/2\). Compute \( A = \) ________
PROBLEMS:

1. If \( a_4 = 1 \), determine \( a_0 \) and \( a_3 \) of the quartic polynomial with the roots: \( 2 - i \) and \( 5 + \sqrt{3} \).

2. If three roots of \( x^4 + Px^2 + Qx + R = 0 \) are \(-6, 5, 6\), what is the value of \( P + R \)?

3. For nonzero constants \( C \) and \( D \), the equation: \( 4x^3 - 12x^2 + Cx + D = 0 \) has two real roots whose sum is \( 0 \). Calculate \( C/D \).

4. The arithmetic mean of \( r \) and \( s \) is \( 6 \) and the geometric mean of \( r \) and \( s \) is \( 10 \). Write a quadratic equation with \( a_2 = 1 \) whose roots are \( r \) and \( s \).

5. If \( r \) and \( s \) are the real roots of \( x^2 + px + 8 = 0 \), then what is the minimum value of \( |r + s| \)?

6. The three roots of \( 64x^3 - 144x^2 + 92x - 15 = 0 \) are in arithmetic progression. What is the positive difference between the largest and the smallest of the three roots?

7. If \( m \) and \( n \) are non-zero roots of \( 5x^2 + 2mx + n = 0 \), then evaluate \( m + n \).

8. Consider \( x^2 + px + q = 0 \), where \( p \) and \( q \) are positive numbers. The roots of this equation differ by \( 1 \). Express \( p \) as a function of \( q \).

9. The sum of the squares of the roots of the equation \( x^2 + 2hx = 3 \) is \( 26 \). Calculate \( h^2 \).

10. The function \( f(x) \) satisfies \( f(3 + x) = f(3 - x) \) for all real numbers \( x \). If the equation \( f(x) \) has exactly four distinct real roots, then what is the sum of these four roots?
VIETA'S FORMULAS

If $a_n = 1$, then the SUM of the roots of $F(x) = r_1 + r_2 + r_3 \ldots = -a_{n-1}$

AND the SUM of the ‘Product Pairs’ of the roots of $F(x) = r_1 r_2 + r_1 r_3 + r_2 r_3 + \ldots = a_{n-2}$

........................................ETC

AND the PRODUCT of the roots of $F(x) = r_1 r_2 r_3 \ldots = (-1)^n a_0$

More generally, for any non-zero value of $a_n$;

then the SUM of the roots of $F(x) = r_1 + r_2 + r_3 \ldots = -a_{n-1} / a_n$

AND the SUM of the ‘Product Pair’s’ of the roots of $F(x) = r_1 r_2 + r_1 r_3 + r_2 r_3 + \ldots = a_{n-2} / a_0 = a_{n-2} / a_n$

........................................ETC

AND the PRODUCT of the roots of $F(x) = r_1 r_2 r_3 \ldots = (-1)^n a_0 / a_n$

ANSWERS:

EXERCISES:

1A. $-7; 9; 5$ 
1B. $9; 4; 3$ 
1C. $0; -10; -2$

2A. $-7/2; 9/2; 5/2$ 
2B. $9/4; -1; -3/4$

3A. $C = 115$ 
3B. $A = -3$

PROBLEMS:

1. $a_3 = -14; \ a_0 = 48$
2. $P + R = 839$
3. $C/D = -1/3$
4. $f(x) = x^2 - 12x + 100$
5. $4\sqrt{2}$
6. $1$
7. $-2/25$
8. $\sqrt{4q + 1}$
9. $h^2 = 5$
10. $12$